A Classical Cosmological Model for Triviality

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The aim of this paper is to study the triviality of $\lambda\phi^4$ theory in a classical gravitational model. Starting from a conformal invariant scalar tensor theory with a self-interaction term $\lambda\phi^4$, we investigate the effect of a conformal symmetry breaking emerging from the gravitational coupling of the large-scale distribution of matter in the universe. Taking in this cosmological symmetry breaking phase the infinite limit of the maximal length (the size of the universe) and the zero limit of the minimal length (the Planck length) implies triviality, i.e. a vanishing coupling constant λ. It suggests that the activity of the self-interaction term $\lambda\phi^4$ in the cosmological context implies that the universe is finite and a minimal fundamental length exists.

1. INTRODUCTION

The simplest renormalized quantum field theory is $\lambda\phi^4$ theory. There are strong evidences that $\lambda\phi^4$ theory is trivial (Bender and Jones, 1988; Deser, 1970; Frohlich, 1982). This feature is generally interpreted to indicate that the interaction in quantum field theory must be sensitive to some cutoff scale. Correspondingly the introduction of a minimal fundamental length as a cutoff length is often considered to be a categorical prerequisite to constructing an interacting theory.

In the present paper, it is argued that triviality of $\lambda\phi^4$ theory can be interpreted as a cosmological effect in a classical gravitational model. It indicates that the concept of triviality can also appear at the classical level. It is also argued that the avoidance of triviality necessiates the introduction of a maximal length in addition to a minimal length. It should be noted that the triviality was derived originally as a quantum effect due to the renormalization group property, so at first sight its interpretation as a classical effect in a gravitational model might seem to be strange. But, we emphasize that there are quantum effects which can be interpreted as classical effects in an appropriate classical model. For example, there is a

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classical model for the confinement (Lehmann and Wu, 1985). This is an effect which was originally discovered as a quantum effect due to the renormalization group property.

The paper is organized as follows: We consider a conformal invariant scalar tensor theory with a self-interaction term $\lambda\phi^4$. A breakdown of conformal invariance is investigated as the effect of the gravitational coupling of the large-scale distribution of matter in the universe. In this cosmological symmetry breaking phase the radius of the universe and the Planck length play the role of the maximal length and the minimal length respectively. We shall study two different limiting procedures. In the first case we allow the maximal length to tend to infinity and in the second case we let the minimal length to tend to zero. It is then argued that both cases lead to triviality. We also present a cosmologioal solution of the theory to justify this result. According to this model the presence of the self-interaction term $\lambda\phi^4$ in the gravitational context implies a finite size for the universe and a non-vanishing minimal length.

2. THE MODEL

We begin with the consideration of a conformal invariant scalar-tensor theory based on the gravitational action

$$S = -\frac{1}{2} \int d^4x \sqrt{-g} \left[g^{\alpha\beta} \nabla_\alpha \phi \nabla_\beta \phi + \frac{1}{6} R\phi^2 - \frac{1}{2}\lambda\phi^4 \right] \tag{1}$$

where ϕ is a scalar field, R is the scalar curvature associated with the metric tensor $g_{\alpha\beta}$ and λ is a dimensionless coupling parameter. Note that this action differs from the standard conformal invariant gravitational action (Levey and Rivers, 1993) by the presence of the self-interaction term $\lambda\phi^4$.

The conformal invariance of the action (1) means that it is invariant under a change in the local unit system, i.e. under conformal transformations

$$g_{\alpha\beta} \to \bar{g}_{\alpha\beta} = \Omega^2(x) g_{\alpha\beta}$$

$$\phi \to \bar{\phi} = \Omega^{-1}(x)\phi \tag{2}$$

in which Ω is a smooth dimensionless space-time function. Therefore all conformal frames, i.e. all local unit systems, are considered to be dynamically equivalent. In practice a particular conformal frame may be singled out by a symmetry breaking effect due to the gravitational coupling of a matter system. We investigate the effect of cosmological symmetry breaking emerging from the gravitational coupling of the large-scale distribution of matter in the universe. In this cosmological symmetry breaking phase the above action is generalized to the action of the scalar-tensor theory

$$S = -\frac{1}{2} \int d^4x \sqrt{-g} \left[g^{\alpha\beta} \nabla_\alpha \phi \nabla_{\beta\phi} + \frac{1}{6} R\phi^2 + \mu^2\phi^2 - \frac{1}{2}\lambda\phi^4 \right] + S_m \tag{3}$$

here S_m stands for the action of the large-scale matter in the universe and μ is a parameter with the dimension of mass. The parameter μ allows us to study breakdown of the conformal invariance in this model. In general, under a conformal transformation all dimensional parameters are required to be transformed according to their dimensions so that μ should obey transformation rule $\mu \to \Omega^{-1}\mu$. The conformal invariance can, however, be broken when a particular conformal frame is chosen in which the dimensional parameter μ takes on a constant configuration. Varying S with respect to $g^{\alpha\beta}$ and ϕ yields

$$G_{\alpha\beta} - 3\mu^2 g\alpha\beta + \tfrac{3}{2}\lambda\phi^2 g_{\alpha\beta} = -6\phi^{-2}(T_{\alpha\beta} + \tau_{\alpha\beta}) \tag{4}$$

$$\Box\phi - \tfrac{1}{6}R\phi + \lambda\phi^3 - \mu^2\phi = 0 \tag{5}$$

where

$$T_{\alpha\beta}(g^{\alpha\beta}) = \frac{-2}{\sqrt{-g}}\frac{\delta}{\delta g^{\alpha\beta}}S_m(g^{\alpha\beta}) \tag{6}$$

and

$$\tau_{\alpha\beta} = \left(\nabla_\alpha\nabla_{\beta\phi} - \frac{1}{2}g_{\alpha\beta}\nabla_{\rho\phi}\nabla^\rho\phi\right) + \frac{1}{6}\left(g_{\alpha\beta}\Box - \nabla_\alpha\nabla_\beta\right)\phi^2 \tag{7}$$

here $T_{\alpha\beta}$ is the matter stress tensor of the universe. Comparing the trace of (4) with the equation (5) yields:

$$g^{\alpha\beta}T_{\alpha\beta} = \mu^2\phi^2 \tag{8}$$

This relation shows that the breakdown of conformal symmetry is related to the large scale distribution of matter in the universe via a non-vanishing trace of $T_{\alpha\beta}$. In this case the length scale μ^{-1} should be related to the typical size of the universe L, namely $\mu^{-1}{\sim}L$. This condition is characteristic to the cosmological symmetry breaking under consideration.

It is possible from this condition to obtain an estimation for the constant backgrbund average value of ϕ which provides the strength of the gravitational coupling. In fact, the trace of $T_{\alpha\beta}$ can be measured in terms of the average density of the large scale distribution of matter, i.e.,

$$T_\alpha^\alpha = \mu^2\phi^2 {\sim} M/L^3 \tag{9}$$

which M denotes the mass of the universe. Now if one uses the empirical fact that the radius of the universe coincides with its Schwarzschild radius $2GM$, one then gets from (9) an estimation of the constant background value of ϕ, namely

$$\phi^2 \sim \frac{M}{L} \sim G^{-1} \sim l_p^{-2} \tag{10}$$

where G and l_p are the gravitational constant and the Planck length respectively.[4] Consequently the gravitational equations (4) reduce to the Einstein field equation with an effective cosmological constant

$$\Lambda_{\text{eff.}} \sim \left(\frac{-1}{L^2} + \frac{\lambda}{l_p^2} \right). \tag{11}$$

This relation implies that in the symmetry breaking phase an effective cosmological constant $\Lambda_{\text{eff.}}$ may be developed as a consequence of two intrinsic cutoff scales, namely the maximal length L and the minimal length l_p. The former cutoff contributes to $\Lambda_{\text{eff.}}$ through the the cosmological symmetry breaking parameter $\mu \sim L^{-1}$ while the latter cutoff contributes to $\Lambda_{\text{eff.}}$ through the coupling constant of the self-interaction term $\lambda \phi^4$.

3. TRIVIALITY

We study the relation (11) with respect to two different limiting procedures, namely the infinite limit of the maximal cutoff and the zero limit of the minimal cutoff. Let us consider the first case, in which the radius L of the universe is ideally taken to infinity. This limit has to be carried out at constant physical conditions in the symmetry breaking phase. Mathematically this means that the symmetry breaking term $\mu^2 \phi^2$ is required to hold constant as $L \to \infty$. By virtue of the relation (9) the last condition may alternatively be expressed by the requirement that the limit $L \to \infty$ has to be carried out at constant matter energy density $\sim \frac{M}{L^3}$. Thus we are applying the infinite-volume limit or the thermodynamic limit of large systems to the universe. Because of $\mu^{-1} \sim L$ this limiting procedure requires through the relations (9)–(10) that the gravitational constant goes to zero. This means that it is interconnected with the the limit of a vanishing minimal length l_p through the cosmological symmetry breaking effect. Thus both intrinsic cutoff scales drop out in this limit.

In order to study the the implication of the limit $L \to \infty$ for the coupling constant λ we use the the equation (11). As $L \to \infty$ the first term of the right hand sight (11) drops out. Since the limit $L \to \infty$ is interconnected with the limit $l_p \to 0$, the relation (11) demands that either the coupling constant λ must go to zero or we must deal with an unacceptable magnification of the effective cosmological constant. Since an infinite cosmological constant requires an infinite vacuum energy density we discard the latter case. The conclusion therefore is that the coupling constant λ must go to zero, that is the

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[4] In this note units are used in which $\hbar = c = 1$.

$\lambda \phi^4$ term disappears. Thus the infinite limit of the maximal length leads to triviality.[5]

Turning now to the second case. we ideally consider the zero limit of the minimal length, that is the Planck length l_p is taken to zero. It follows from the relations (9)–(10) that this limit is interconnected with the limit of the infinite radius of the universe at constant matter energy density. Therefore triviality in this case can be established by the same argument as presented in the first case.

This consideration indicates that triviality can be related to two limiting procedures, the infinite limit of the maximal length and the zero limit of the minimal length. Both limiting procedures are mutually connected through the relations (9)–(10). It should be noted that this triviality argument was presented on the basis of the relations (9)–(10) which holds in the conformal frame singled out by the cosmological symmetry breaking effect we considered, namely the conformal frame in which μ (or equivalently ϕ) is constant.

4. COSMOLOGICAL SOLUTION

In this section we attempt to obtain a cosmological solution for the classical theory under consideration to justify the result of the previous sections. We consider the homogeneous and isotropic cosmological models in the interacting scalar tensor theory of gravity (3). Accordingly, we start with the Robertson–Walker line element and the energy tensor of a dust as the matter

$$ds^2 = dt^2 - S^2(t)\left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\right]$$

$$g^{\alpha\beta} T_{\alpha\beta} = \rho \tag{12}$$

On account of the space time symmetry, we regard the scalar field ϕ as a function of the cosmic time only. We introduce the quantities $H(t)$ and $F(t)$ through the relations

$$H = \dot{S}/S \quad F = \dot{\phi}/\phi \tag{13}$$

where the overdot implies derivative with respect to t, cosmic time. From the coefficient of the energy momentum tensor $T_{\alpha\beta}$ in Eq.(4) one can postulate that the gravitational coupling G behaves as

$$G \sim \phi^{-2}. \tag{14}$$

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[5] This argument applies also if we simply assume that the effective cosmological constant is zero as $L \to \infty$. But this assumption is probably not correct if the infinite limit is carried out at a constant matter energy density.

This assumption implies $\mu^2 = \rho G$, through the Eq.(8). With the above definitions, equations (4) and (5) lead to two independent equations

$$H^2 + k/S^2 - \lambda/2G = \rho G + F^2 + 2HF$$
$$k/S^2 - \dot{H} = 3\rho G + F^2 - \dot{F} + HF \qquad (15)$$

Now, we impose $k = 1$ to assure a finite universe and proceed to present a solution to describe the universe in the present epoch. Thus, we consider the case $F = 0$ or $G = $ constant which also implies the breakdown of conformal symmetry through choosing a constant configuration for ϕ. Also, to impose the characteristics of the present epoch we approximate H with H_0 in which H_0 is the Hubble constant at the present epoch. Under these considerations one can obtain the following constraint on λ and ρ

$$\lambda = 4\rho G^2 + 2H_0^2 G$$
$$3\rho G = 1/S^2 \qquad (16)$$

In a closed universe, the radius of the universe is proportional to the scale factor, $L \sim S$. Applying $G \sim l_p^2$ provides the following relations

$$\mu \sim L^{-1}, \quad MG \sim L$$
$$\Lambda_{\text{eff}} \sim -\frac{1}{L^2} + \frac{\lambda}{l_p^2} \qquad (17)$$

which are analogous to the equations (10) and (11). Therefore, investigating this model in limit $l_p \to 0$ or $L \to \infty$ at a fixed time, can establish the triviality by the same argument as presented in the third section. On the other hand, it can be. seen from equation (17) that the exact dynamical solution of this theory determines the configuration of λ and the triviality also directly can be obtained from this equation in the above limiting procedure. It should be noted that, the limiting procedure at a fixed time cannot be a dynamical limit, and is just an infinite volume limit or thermodynamic limit of large systems.

5. CONCLUDING REMARKS

The basic result of this paper is that the gravitational coupling of the large-scale matter predicts a maximal cutoff length, namely the size of the universe, to prevent the triviality of $\lambda\phi^4$ theory in a classical gravitational model. This means that the triviality of the self-interacting term $\lambda\phi^4$ can be avoided by the cosmological characteristics of a finite universe.

This is a result which has not appeared in the conventional approach to triviality that has been established so far, where the gravitational coupling of the large-scale matter is ignored and only a minimal cutoff is suggested for the construction of an interacting theory in quantum field theory. In fact, this cosmological

aspect of triviality indicates that the minimal cutoff length and the maximal cutoff length may be considered as being basically interconnected by the conditions of an interacting theory in the gravitational context.

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